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Optimal Control for Parabolic Equations[1]

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1. INTRODUCTION

Let B denote a bounded domain in R^n and set $D_T = B \times [0, T)$ for $T > 0$. We denote by x a variable point in B and by (x, t) a variable point in D_T. Set $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = \partial/\partial x_j$, $\alpha = (\alpha_1, ..., \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Consider the differential operator

$$Lu \equiv \frac{\partial u}{\partial t} - A(x, t, D)\, u \equiv \frac{\partial u}{\partial t} - \sum_{|\alpha| \leqslant 2m} a_\alpha(x, t)\, D^\alpha u, \tag{1.1}$$

where m is a positive integer. Throughout this work we assume that the coefficients $a_\alpha(x, t)$ are sufficiently smooth in $\bar{D}_\infty$, that the boundary ∂B of B is sufficiently smooth, and that L is parabolic in the sense of Petrowski in $\bar{D}_\infty$ (see, for instance, [1]).

Consider the initial-boundary value problem

$$Lu = f(x, t) \qquad \text{in some} \qquad D_{t_0}, \tag{1.2}$$

$$u(x, 0) = \varphi(x) \qquad \text{on} \qquad B, \tag{1.3}$$

$$B_j(x, t, D)\, u = p_j(x, t) \qquad \text{on} \qquad \partial B \times (0, t_0) \qquad (1 \leqslant j \leqslant m), \tag{1.4}$$

where the B_j are some "regular" boundary operators (such that the problem (1.2)-(1.4) has a unique smooth solution); for instance, $B_j = \partial^{j-1}/\partial \nu^{j-1}$ where ν is the outward normal to ∂B. Set $p = (p_1, ..., p_m)$.

We are interested in the situation where all but one of the functions f, φ, p are fixed, and the one which is not fixed (called the *control*) is subject to the condition of belonging to a certain set (called the *control set*). We then consider the following problem: Given a set W (called the *target set*), say in some $L^s(B)$, find a control such that the corresponding solution reaches W at the smallest possible time.

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This problem can be classified as a time-optimal problem with a given target W. In a previous paper [2] we have considered this problem for the evolution equation $du/dt + Au = f$, with a given $u(0)$ and with control f, in Banach space. There, the target consisted of one point. In the present work we treat problems where the control can also be on the boundary, but W is a convex body. Our main object is to prove a Bang-bang principle, but for the sake of completeness we also prove the existence of a minimum. The smoothness of the optimal control is also derived.

In Section 2 we consider the case where the control is on the lateral boundary. In Section 3 we consider the case where the control is on the right-hand side of (1.2). Finally, in Section 4 we consider the case of boundary control when the end-point is completely free; the problem is to minimize a given functional at a given time.

Time-optimal problems with control on the boundary have been treated by Egorov [3], [4]. He proved a Bang-bang principle in the special case where L is the heat operator in one dimension and W is the unit ball in L^2. His proof can be extended to the case where $A(x, t, D)$ is a second-order elliptic operator with coefficients analytic and independent of t, provided the boundary and boundary conditions are analytic, and provided the space dimension is $\leqslant 3$ (the last restriction follows from the need to use the theorem of Müntz); W can be any convex set in L^2, satisfying the condition (A) of Section 2. The method of the present paper is much more general than that of Egorov.

For simplicity we assume that all the functions in this paper are real valued.

2. Control on the Boundary

Here f, φ are fixed and $p = (p_1, ..., p_m)$ is the control. For simplicity we consider first the case of second-order parabolic equations, i.e., $m = 1$. We take the boundary condition (1.4) to be

$$\frac{\partial u}{\partial \mu} + a(x, t)\, u = p(x, t) \qquad \text{on} \qquad \partial B \times (0, t_0), \tag{2.1}$$

where $a(x, t)$ is a smooth function $\geqslant 0$ and $\partial/\partial\mu$ is the outward transversal derivative on the lateral boundary (see, for instance [1]).

Denote by $G(x, t; \xi, \tau)$ the Green function for the problem (1.2), (1.3), (2.1), i.e., $G(x, t; \xi, \tau)$ is a fundamental solution satisfying $L^*G = 0$ and (2.1), as a function of (ξ, τ). We can rewrite the solution of (1.2), (1.3), (2.1), for smooth f, φ, p, in the form

$$u(x, t) = \beta(x, t) + \int_0^t \int_{\partial B} G(x, t; \xi, \tau)\, p(\xi, \tau)\, dS_\xi\, d\tau, \tag{2.2}$$

where $\beta(x, t)$ is a fixed smooth function. We have the bound:

$$| G(x, t; \xi, \tau) | \leqslant \frac{C}{(t-\tau)^{n/2}} \exp \left\{ -c \frac{| x - \xi |^2}{t - \tau} \right\} \qquad (C > 0, c > 0). \tag{2.3}$$

LEMMA 1. *Let $p(\xi, \tau)$ be a measurable function on $\partial B \times (0, T)$ such that $\| p(\cdot, \tau) \|_{L^q(\partial B)}$ is a bounded function of τ, and $q > 1$. Let $K(x, t; \xi, \tau)$ be a continuous function in all its variables ($x \in \bar{B}$, $\xi \in \bar{B}$, $0 \leqslant \tau < t \leqslant T$) satisfying*

$$| K(x, t; \xi, \tau) | \leqslant \frac{C}{(t-\tau)^{(n+2m-i)/2m}} \exp \left\{ -c \left[\frac{| x - \xi |^{2m}}{t - \tau} \right]^{1/(2m-1)} \right\}, \tag{2.4}$$

where m is a positive integer and $1 \leqslant i \leqslant 2m$. If $q > (n-1)/(i-1)$, then the integral

$$\int_0^t \int_{\partial B} K(x, t; \xi, \tau) p(\xi, \tau) \, dS_\xi \, d\tau \tag{2.5}$$

belongs to $L^\infty(B)$ for each $0 < t \leqslant T$, and it is a continuous function from $(0, T]$ into $L^\lambda(B)$, for any $1 \leqslant \lambda < \infty$.

PROOF. We may restrict x to be in a sufficiently small neighborhood of ∂B and restrict ξ to be in some neighborhood of x. Using (2.4) we then find that it suffices to estimate the $L^\infty(B' \times I)$-norm of

$$F(x', s) = \int_0^t \int_{B'} \frac{C}{(t-\tau)^{(n+2m-i)/2m}} \exp \left\{ -c \left[\frac{s^{2m}}{t-\tau} \right]^{1/(2m-1)} \right\}$$
$$\times \exp \left\{ -c \left[\frac{| x' - \xi' |^{2m}}{t - \tau} \right]^{1/(2m-1)} \right\} | p'(\xi', \tau) | \, d\xi' \, d\tau,$$

where B' is some $(n-1)$-dimensional bounded domain, I is an interval $0 \leqslant s \leqslant s_0$, and $p'(\xi, \tau)$ satisfies: $\| p'(\cdot, \tau) \|_{L^q(B')} \leqslant$ const.

It clearly suffices to estimate $F(x', 0)$. To estimate the inner integral of $F(x', 0)$, we use Hölder's inequality with exponents q, r $(1/q + 1/r = 1)$ and then substitute $\rho = | x' - \xi' | / (t - \tau)^{1/2m}$. We find that $F(x', 0)$ is a bounded function provided

$$\left(\frac{n + 2m - i}{2m} r - \frac{n-1}{2m} \right) \frac{1}{r} < 1.$$

But this inequality is a consequence of our assumptions on q.

The assertion concerning the continuity of the integral (2.5) now also follows, by a standard argument.

Lemma 1 is not applicable in case $i = 1$. The following lemma clearly deals also with this case.

LEMMA 1′. *Let p, K be as in Lemma 1, with $q > 1$, $q > (2m + n)/i$. Then the integral (2.5) belongs to $L^r(B)$ for every $0 < t \leq T$ (where $1/q + 1/r = 1$), and it is a continuous function from $(0, T]$ into $L^r(B)$.*

PROOF. It is enough to prove that

$$\int_0^{s_0} | F(x', s) |^r \, ds \leq C_1 \qquad (x' \in B', C_1 \text{ constant}).$$

By using Hölder's inequality for the double integral of F, with exponents q, r, and substituting $\rho = | x' - \xi' |/(t - \tau)^{1/2m}$, the assertion follows provided

$$\frac{n + 2m - i}{2m} r - \frac{n}{2m} < 1.$$

But this inequality is a consequence of our assumptions on q.

Formula (2.2) gives the solution of (1.2), (1.3), (2.1) for smooth $p(\xi, \tau)$. We now define the solution of (1.2), (1.3), (2.1) for any integrable function $p(\xi, \tau)$ to be the right-hand side of (2.2). In view of Lemmas 1, 1′, if $p(\cdot, \tau) \in L^q(\partial B)$ $(q > 1)$ and its norm is a bounded function, then the solution $u(x, t)$ belongs to $L^s(B)$ for each fixed $t > 0$, where $1 \leq s \leq \infty$ if $q > n - 1$ and $1 \leq s < (n + 2)/2$ if $(n - 2)/2 < q \leq n - 1$.

Any measurable function $p(\xi, \tau)$ on $\partial B \times (0, T)$ for which $\| p(\cdot, \tau) \|_{L^q(\partial B)}$ is a bounded function for some $1 < q \leq \infty$, will be called a *control function.* We now fix such a number q $(1 < q \leq \infty)$, and a subset U in $L^q(\partial B)$. We call U the *control set.* A control $p(\xi, \tau)$ is called *admissible* if $p(\cdot, \tau) \in U$ for almost all τ and if $\| p(\cdot, \tau) \|_{L^q(\partial B)}$ is a bounded function. When U is a bounded set, the last condition is superfluous.

We fix a number s such that $1 \leq s < \infty$ if $q > n - 1$ and $1 \leq s < (n + 2)/2$ if $(n + 2)/2 < q \leq n - 1$.

Let W be a fixed set in $L^s(B)$. We call W the *target set.* The following assumption will be needed:

(A) W is a closed convex set in $L^s(B)$, with nonzero interior. At each point $z \in \partial W$ there exists a unique tangent hyperplane $\Pi(z)$ (determined by a continuous linear functional), and, for each point y lying in the interior of the same half-space with respect to $\Pi(z)$ in which W lies, the interval (y, z) contains an interior point of W.

Condition (A) roughly states that W is a convex body with no angular points.

Throughout this section it is always assumed that there exists an admissible control $p(\xi, \tau)$ $(0 \leq \tau \leq t_0)$ such that the corresponding solution $u(x, t)$ of (1.2), (1.3), (2.1), as given by (2.2), satisfies: $u(\cdot, t_0) \in W$.

Denote by T the infimum of all the t_0's as above. If there exists an admissible control $p^*(\xi, \tau)$ $(0 \leqslant \tau \leqslant T)$ such that the corresponding solution $u^*(x, t)$ satisfies: $u^*(\cdot, T) \in W$, then we call p^*, u^* a *time-optimal solution.*

THEOREM 1. *If U is a bounded, closed, convex set and W is a closed convex set, then there exists a time-optimal solution.*

PROOF. Take a minimizing sequence $\{p^{(i)}, u^{(i)}\}$ with $u^i(\cdot, t_i) \in W$, $t_i \searrow T$, where T is the optimal time. $\{p^{(i)}\}$ is a bounded sequence in $L^2((0, t_1); L^q(\partial B))$ (if we extend each $p^{(i)}$ by zero for $t > t_i$). We can therefore extract a subsequence (denote it again by $\{p^{(i)}\}$) which is weakly convergent to some p^*. Then $p^*(\xi, \tau)$ belongs to $L^2((0, T); L^q(\partial B))$. By an argument used in the proof of Theorem 4 of [2] we conclude that $p^*(\cdot, \tau) \in U$ for almost all τ, i.e., p^* is an admissible control.

Denote by u^* the solution of (1.2), (1.3), (2.1) corresponding to p^*. By the Ascoli-Arzela Lemma and by the weak convergence of $p^{(i)}$ to p^*, it follows that $u^{(i)}(x, t_i) \to u^*(x, T)$ uniformly on compact subsets of B (actually one should take a subsequence). From the proof of Lemmas 1, 1′ we also conclude that $\{u^{(i)}(\cdot, t_i)\}$ is a bounded sequence in $L^s(B)$. Combining these two facts, it follows that $\{u^i(\cdot, t_i)\}$ is weakly convergent in $L^s(B)$ to $u^*(\cdot, T)$. Since each element of this sequence is in W, and since W is a closed convex set, it follows that $u^*(\cdot, T) \in W$. This completes the proof.

We shall need the following uniqueness property.

(B) Let L^* denote the adjoint of L. For any $T > 0$ and for any smooth function $v(x, t)$ in $\bar{D}_T$ satisfying: $L^*v = 0$ in D_T, $\partial v/\partial \mu + av = 0$ on $\partial B \times (0, T)$, if $v(x, t) = 0$ on $\partial B \times \Delta$ where Δ is some subset of $(0, t)$ of positive measure, then $v(x, T) \equiv 0$.

LEMMA 2. *If $a(x, t)$ and the coefficients of L are analytic functions, and if ∂B is an analytic manifold, then* (B) *holds.*

PROOF. Let s be a density point of Δ and take any angular domain on $\partial B \times (0, T)$ with vertex (ξ, s), for some fixed $\xi \in \partial B$. Since v vanishes on a sequence of points of this angular domain, we conclude, by Rolle's theorem, that a certain derivative of v at (ξ, s), in a direction lying in the angular domain, must vanish. Since the opening of the angular domain can be arbitrarily small, all the first tangential derivatives of v vanish at (ξ, s). Since almost all the points of Δ are regular points, and since v is C^∞ in $\bar{D}_T$, we find that all the first tangential derivatives of v vanish on $\partial B \times \Delta$.

Repeating the argument step by step, we conclude that the restriction of $v(x, t)$ to $\partial B \times (0, T)$ vanishes to any order on $\partial B \times \Delta$. Since, by [5], this function is analytic, it follows that $v \equiv 0$ on $\partial B \times (0, T)$. From the relation $\partial v/\partial \mu + av = 0$ we conclude that $\partial v/\partial \nu \equiv 0$ on $\partial B \times (0, T)$. Using the Cauchy-Kowalewski theorem and the fact (see [5]) that $v(x, t)$ is analytic in D_T, we find that $v \equiv 0$ in some neighborhood of $\partial B \times (0, T)$. It follows that $v \equiv 0$ in D_T.

THEOREM 2. *Let U be a convex set, let W satisfy* (A), *and let* (B) *hold. If $p^*(\xi, t)$, $u^*(x, t)$ is a time-optimal solution with time T, then $p^*(\cdot, t) \in \partial U$ for almost all t, $0 \leqslant t \leqslant T$.*

Theorem 2 is an example of a Bang-bang principle. As in [2], we can deduce from Theorem 2:

COROLLARY 1. *If U is strictly convex then there exists at most one time-optimal solution.*

In proving this corollary we use the fact that the arithmetic mean of two optimal solutions is an optimal solution (the convexity of W is hereby employed).

PROOF OF THEOREM 2. Consider the set

$$\Omega_T = \Big\{ v \in L^s(B), v(x) = \beta(x, T) + \int_0^T \int_{\partial B} G(x, T; \xi, \tau)\, p(\xi, \tau)\, dS_\xi\, d\tau,$$
$$p \text{ any admissible control} \Big\}.$$

Since U is a convex set, also Ω_T is convex. The set W is also convex and $W \cap \Omega_T$ contains the point $z = u^*(\cdot, T)$. This point must lie on the boundary of W. Indeed, if $z \in \operatorname{int} W$ then, by Lemmas 1, 1′, $u^*(\cdot, T - \epsilon)$ lies in W for all $\epsilon > 0$ sufficiently small. This however contradicts the minimality of T.

Denote by $\Pi(z)$ the tangent hyperplane to W at z and by $\Pi_-(z)$, $\Pi_+(z)$ the closed half spaces determined by $\Pi(z)$ with $W \subset \Pi_+(z)$. We claim that $\Omega_T \subset \Pi_-(z)$. Indeed, otherwise there exists a point $y \in \Omega_T$ which lies in the interior of $\Pi_+(z)$. By the assumption (A), there exists an interior point $\bar{z}$ of W lying in the interval (z, y). Since $\bar{z}$ must also lie in Ω_T, we obtain a contradiction to the minimality of T.

Let $\gamma(x)$ be the nonzero element in $L^{s'}(B)$ (where $1/s + 1/s' = 1$) which determines the hyperplane $\Pi(z)$. Then

$$\int_B v(x)\, \gamma(x)\, dx \leqslant \int_B u^*(x, T)\, \gamma(x)\, dx \qquad \text{for any} \qquad v \in \Omega_T. \tag{2.6}$$

Suppose now that the assertion of the theorem is false. Then there exists a subset Δ of $(0, T)$ having positive measure, such that

$$\text{dist.}\,(p^*(\cdot, t), \partial U) \geqslant \delta > 0 \qquad \text{for all} \qquad t \in \Delta.$$

Hence for every bounded measurable function $w(\xi, \tau)$ on $\partial B \times (0, T)$ with support in $\partial B \times \Delta$, the function $p^* + \epsilon w$ is an admissible control for all real ϵ with $|\epsilon|$ sufficiently small. Substituting in (2.6) the v corresponding to $p^* + \epsilon w$, we easily conclude that

$$\int_{\Delta}\int_{\partial B}\left\{\int_{B} G(x, T; \xi, \tau)\,\gamma(x)\,dx\right\} w(\xi, \tau)\,dS_\xi\,d\tau = 0.$$

Since w is arbitrary, it follows that the function

$$\Gamma(\xi, \tau) \equiv \int_B G(x, T; \xi, \tau)\,\gamma(x)\,dx \tag{2.7}$$

vanishes on $\partial B \times \Delta$. Applying the assumption (B) with $v = \Gamma$, we get $\Gamma(\xi, \tau) \equiv 0$ for $\xi \in B$ and each τ sufficiently close to T. Taking $\tau \nearrow T$ we obtain $\gamma(x) = 0$ for almost all $x \in B$, which is impossible.

From (2.6) we also obtain some further information on the form of $p^*(\xi, \tau)$. In fact, (2.6) is equivalent to

$$\int_0^T \int_{\partial B} \Gamma(\xi, \tau)\,p(\xi, \tau)\,dS_\xi\,d\tau \leqslant \int_0^T \int_{\partial B} \Gamma(\xi, \tau)\,p^*(\xi, \tau)\,dS_\xi\,d\tau \tag{2.8}$$

for any admissible control p. Note that (2.8) is an analog of Pontryagin's maximum principle. Since $\Gamma(\cdot, \tau) \neq 0$ for almost all τ ($\Gamma(\cdot, \tau)$ is considered here as an element of $L^q(\partial B)$), we get:

Corollary 2. *If U is the unit ball in $L^q(\partial B)$ then*

$$p^*(\xi, \tau) = \frac{\Gamma(\xi, \tau)}{\|\Gamma(\cdot, \tau)\|_{L^q(\partial B)}} \quad \textit{almost everywhere.} \tag{2.9}$$

Thus, in particular, $p^*(\xi, \tau)$ can be taken as a smooth function (by modifying it on a set of measure zero).

This corollary obviously extends to other convex sets U. (2.9) has to be replaced by a more involved formula depending on the geometry of U. One can also impose a geometric condition on U under which $p^*(\xi, \tau)$ can be taken to be a smooth function.

Generalizations. (I) Theorems 1, 2 remain true if the transversal derivative $\partial/\partial\mu$ in (2.1) is replaced by any nontangential outward oblique derivative $\partial/\partial\lambda$. The condition (B) needs to be slightly modified, but Lemma 2 remains true for this modified condition.

(II) Theorems 1, 2 extend without difficulty to the general system (1.2)-(1.4). First we represent the solution in the form

$$u(x, t) = \beta(x, t) + \sum_{j=1}^{m} \int_0^t \int_{\partial B} G_j(x, t; \xi, \tau)\,p_j(\xi, \tau)\,dS_\xi\,d\tau \tag{2.10}$$

where, say,

$$|G_j(x, t; \xi, \tau)| \leqslant \frac{C}{(t-\tau)^{(n-2m-i_j)/2m}} \exp\left\{-c\left[\frac{|x-\xi|^{2m}}{t-\tau}\right]^{1/(2m-1)}\right\}. \quad (2.11)$$

We take $p_j(\cdot, t) \in U_j$ where $U_j \subset L^{q_j}(\partial B)$, and set $p = (p_1, ..., p_m)$, $U = U_1 \times \cdots \times U_m$. Let s be any integer such that the following holds for each j:

$$1 \leqslant s < \infty \qquad \text{if} \qquad q_j > \frac{n-1}{i_j - 1}$$

and

$$1 \leqslant s < \frac{2m+n}{2m+n-i_j} \qquad \text{if} \qquad \frac{2m+n}{i_j} < q_j \leqslant \frac{n-1}{i_j-1}.$$

In view of Lemmas 1, 1′, all the integrals in (2.10) belong to $L^s(B)$. The target set W is taken in $L^s(B)$.

One can now proceed as in the proofs of Theorems 1, 2. In generalizing the proof of Theorem 2, we need a condition similar to the condition (B), or, actually, just the following:

(B′) Let $\gamma(x) \in L^{s'}(B)$ and let $F_j(\xi, \tau) \equiv \int_B \gamma(x)\, G_j(x, T; \xi, \tau)\, dx$ vanish on $\partial B \times \Delta$ for $j = 1, ..., m$, where Δ has a positive measure. Then $\gamma(x) = 0$ almost everywhere.

This condition can be verified for some systems by extending the proof of Lemma 2. Consider, for example, the case where $A(x, t, D) = \Delta^2$ (Δ = Laplacian) and the conditions in (1.4) are the Dirichlet conditions: $u = \partial u/\partial\nu = 0$. From Green's formula we get $G_1 = \Delta_\xi G$, $G_2 = -\partial(\Delta_\xi G)/\partial\nu$, where G is Green's function. Setting

$$\Gamma(\xi, \tau) = \int_B \gamma(x)\, G(x, T; \xi, \tau)\, dx,$$

we then have: $\Gamma = \partial\Gamma/\partial\nu = 0$ on $\partial B \times (0, T)$ and $\Delta\Gamma = \partial(\Delta\Gamma)/\partial\nu = 0$ on $\partial B \times \Delta$. Writing $\Delta = \partial^2/\partial\nu^2 + \sum_j \partial^2/\partial s_j^2$, s_j tangential directions to ∂B, and using the method of proof of Lemma 2, we get $\partial^2\Gamma/\partial\nu^2 = 0$, $\partial^3\Gamma/\partial\nu^3 = 0$ on $\partial B \times (0, T)$ provided ∂B is analytic. We can now proceed, as in the proof of Lemma 2, and prove that $\Gamma \equiv 0$ for $\xi \in B$, $0 \leqslant \tau < T$. Hence $\gamma(x) = 0$ almost everywhere.

The above considerations extend to the parabolic system with $A(x, t, D) = \Delta^m$, $B_j = \partial^{j-1}/\partial\nu^{j-1}$, where m is any positive integer. Thus, in particular, Theorem 2 holds for this system provided ∂B is analytic (and the condition (B) omitted).

We finally point out that the set $U = U_1 \times \cdots \times U_m$ is not strictly convex if $m > 1$ (even if all its components are strictly convex).

(III) All the previous results remain true if the control $p(\xi, \tau)$ is not free on the whole lateral boundary $\partial B \times (0, T)$, but only on a part of it, say, on $\partial_0 B \times (0, T)$, where $\partial_0 B$ is a portion of ∂B. The proofs remain essentially unchanged.

3. Control in the Domain

In this section the term $f(x, t)$ occurring in (1.2) is the control function, whereas the p_j and φ are fixed. With the aid of Green's function, we can write the solution of (1.2)-(1.4) in the form

$$u(x, t) = \beta_0(x, t) + \int_0^t \int_B G(x, t; \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau, \tag{3.1}$$

where β_0 is a given smooth function. We assume that the boundary operators B_j are "regular" in the sense, say, that G exists and (2.11) holds with $G_j = G$ and $i_j = 2m$.

Suppose that $f(\xi, \tau)$ is a measurable function in D_T such that, for some $1 < q \leqslant \infty$,

$$\|f(\cdot, \tau)\|_{L^q(B)}$$

is a bounded function of τ. We call f a *control function.* By Young's inequality it follows that the integral in (3.1) belongs to $L^s(B)$ if $1 \leqslant s \leqslant q$. By modifying the proofs of Lemmas 1, 1′ we also find that if $q > n/2m$ then this integral is in $L^\infty(B)$.

We now fix q, $1 < q \leqslant \infty$, and a *control set* U in $L^q(B)$ and call f an *admissible control* if $f(\cdot, \tau) \in U$ for almost all τ. We also fix a *target set* W in some space $L^s(B)$ where $1 \leqslant s < \infty$ if $q > n/2m$ and $1 \leqslant s \leqslant q$ if $q \leqslant n/2m$.

It is assumed throughout this section that there exists an admissible control $f(\xi, \tau)$ $(0 < \tau < t_0)$ with $u(x, t_0)$ in W. We define a time-optimal problem (with respect to the target W) in the obvious way.

Theorem 3. *If U is a bounded, closed, convex set and W is a closed convex set, then there exists a time-optimal solution.*

The proof is similar to that of Theorem 1.

We next consider the question of uniqueness. We introduce the set

$$\Omega_T{}^0 = \Big\{ v \in L^s(B);\, v(x) = \beta_0(x, T) + \int_0^T \int_B G(x, t; \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau,$$

$$f \text{ any admissible control} \Big\}.$$

This is a convex set in $L^s(B)$. Suppose that f^*, u^* is an optimal solution. Then $u^*(\cdot, T)$ lies in the intersection $\Omega_T{}^0 \cap W$. Proceeding analogously to Section 2, we get

$$\int_0^T \int_B \Gamma_0(\xi, \tau) f(\xi, \tau)\, d\xi\, d\tau \leqslant \int_0^T \int_B \Gamma_0(\xi, \tau) f^*(\xi, \tau)\, d\xi\, d\tau \tag{3.2}$$

for any admissible control f, where

$$\Gamma_0(\xi, \tau) = \int_B G(x, T; \xi, \tau)\, \gamma_0(x)\, dx \tag{3.3}$$

and γ_0 is a nonzero function in $L^{s'}(B)$, $1/s + 1/s' = 1$.

If

$$\text{dist.}\,(f^*(\cdot, \tau), \partial U) \geqslant \delta > 0 \qquad \text{for all} \qquad t \in \Delta,$$

where Δ is a subset of $(0, T)$ having positive measure, then we find, upon using (3.3) (compare Section 2), that $\Gamma_0(x, t) \equiv 0$ on $B \times \Delta$. Hence, if A^* (the adjoint of the elliptic operator A defined by the operator $A(x, t, D)$ of (1.1) and the homogeneous conditions $B_j u = 0$) satisfies the weak backward uniqueness property (as defined in [2]), then $\Gamma(x, t) \equiv 0$ for each t sufficiently close to T. Taking $t \nearrow T$ we get $\gamma_0(x) = 0$ almost everywhere, which is impossible.

We have thus proved:

THEOREM 4. *Let U be a convex set and let W satisfy* (A). *Assume that A^* satisfies the weak backward uniqueness property. If $f^*(\xi, \tau)$, $u^*(\xi, \tau)$ is a time-optimal solution with time T, then $f^*(\cdot, \tau) \in \partial U$ for almost all t, $0 \leqslant t \leqslant T$.*

Corollaries 1, 2 of Theorem 2 can obviously be extended to the present case.

GENERALIZATIONS. (I) Theorems 3, 4 extend to the case where the control $f(\xi, \tau)$ is free only on a portion of the domain, say, on $B_0 \times (0, T)$, where B_0 is a subdomain of B. In extending Theorem 4, we need a stronger backward uniqueness property. If the coefficients of L are analytic functions, then the weak backward uniqueness property is sufficient.

(II) Theorems 3, 4 can be extended to evolution equations

$$\frac{du}{dt} + A(t)\, u = f(t) \tag{3.4}$$

in a Banach space X. $A(t)$ is assumed to be such that a strongly continuous fundamental solution $S(t, \tau)$ exists. The solution of (3.4) with the initial condition

$$u(0) = u_0 , \tag{3.5}$$

is then given by

$$u(t) = S(t, 0)\, u_0 + \int_0^t S(t, \tau) f(\tau)\, d\tau. \tag{3.6}$$

The control set U and the target set W are taken to be convex sets in X.

The existence of a time-optimal solution is proved as in [2] (where W consists of one point). To prove a Bang-bang principle, let $f^*(t)$, $u^*(t)$ be an optimal solution with time T. By arguments as in Section 2 we can prove that if W satisfies (A) (with $L^s(B)$ replaced by X) then there exists a supporting functional to Ω_T at $u^*(T)$. We can then proceed as in [2] and show:

(a) If $A(t)$ is independent of t then $f^*(t) \in \partial U$ for almost all t.

(b) If $A(t)$ depends on t, if X is a Hilbert space, and if $A^*(t)$ satisfies the weak backward uniqueness property, then $f^*(t) \in \partial U$ for almost all t.

Corollary 1 of Theorem 4 can obviously be extended to the present case. In case (b) holds, also Corollary 2 can be extended.

(III) One can prove uniqueness theorems also if U is not strictly convex. It will be enough to do it for (3.4) in a Hilbert space X. Assume that U is bounded and closed, and that in every direction q there exists a supporting hyperplane Π_q to U orthogonal to q. Assume also that for at most a sequence of q's, say $\{q_m\}$, $\Pi_{q_m} \cap U$ consists of more than one point. Finally, assume that A, U are in *general position* in the sense that the q_m are not eigenfunctions of A^*.

THEOREM 5. *Let the foregoing assumptions hold and assume that A is independent of t and has the backward uniqueness property, and that $S(t)$ is analytic. If W satisfies (A) (with respect to X) and is strictly convex, then there exists a unique optimal solution.*

PROOF. Suppose $f_0(t)$, $u_0(t)$ and $f_1(t)$, $u_1(t)$ are optimal solutions. Then the same is true of their arithmetic mean, so that $u_0(T) = u_1(T)$. Let p_0 be the supporting functional to Ω_T at $u_0(T)$. Then

$$\int_0^T (S^*(T-\tau)p_0 , f_0(\tau))\, d\tau \geqslant \int_0^T (S^*(T-\tau)p_0 , v(\tau))\, d\tau$$

for every admissible control $v(\tau)$. It then easily follows that for almost all τ,

$$(S^*(T-\tau)p_0 , f_0(\tau)) \geqslant (S^*(T-\tau)p_0 , v)$$

for all $v \in U$, and equality holds for $v = f_1(\tau)$.

Hence $f_0(\tau)$ and $f_1(\tau)$ lie on Π_q where $q = S^*(T - \tau)p_0/\| S^*(T - \tau)p_0 \|$. If now $f_0(\tau) \neq f_1(\tau)$ on a set of positive measure, then there exists a $q = q_m$ such that

$$S^*(T - \tau)p_0 = \| S^*(T - \tau)p_0 \| q_m \tag{3.7}$$

for a sequence of τ's. But then (3.7) holds identically. Using the semi-group property of $S(t)$, we find that the right hand side of (3.7) has the form $Ce^{-\gamma(T-\tau)}q_m$ and (by taking $\tau = T$) $Cq_m = p_0$. Thus

$$e^{\gamma(T-t)}S^*(T - t)q_m = q_m .$$

This implies $A^*q_m = \gamma q_m$ which is impossible.

REMARK 1. It is interesting to compare the Bang-bang results of [2] with the present ones. In [2] the coefficients of the equation are independent of t and the end-point is fixed, whereas here the coefficients may depend on t but the end-point is free to move on the boundary of a convex body W. We also derive here some formulas for the optimal control, such as (2.9). (In [2] we derived such formulas only in case A is (roughly) assumed to generate a strongly-continuous group.)

REMARK 2. We shall not treat here the case where the control function is the initial data φ. This case can easily be treated by the present method, and the results are analogous to Theorems 3, 4.

4. FREE END-POINT WITH CONTROL ON THE BOUNDARY

We wish to minimize a certain functional at a fixed time T, when the control is given on the boundary. If the control is the nonhomogeneous term f occurring in (1.2), then the problem was already considered in [2]. That method also applies to the present problem of boundary control. We shall therefore make the discussion brief and, moreover, consider only one example of a cost-functional, namely,

$$I(u) = \int_B | u(x, T) - u_0(x) |^2 \, dx. \tag{4.1}$$

Here u is the solution of (1.2)-(1.4) when f, φ are fixed, and $p = (p_1, \ldots, p_m)$ is the control. u is given by (2.10). For simplicity we take $f \equiv 0$, $\varphi \equiv 0$.

We use the notation following (2.10) and assume that $s = 2$ satisfies the inequalities imposed there. Then $u(x, t)$ is in $L^2(B)$ for every fixed $t > 0$.

Let p^*, u^* be an optimal control, and define

$$\bar{p}(\xi, \tau) = \begin{cases} p^*(\xi, \tau) & \text{if} \quad \tau \notin (\tau_1, \tau_1 + \epsilon) \\ g(\xi) & \text{if} \quad \tau \in (\tau_1, \tau_1 + \epsilon), \end{cases}$$

for $\epsilon > 0$ sufficiently small, where τ_1 is a regular point of $p^*(\cdot, \tau)$. Let $\bar{u}$ be the solution corresponding to $\bar{p}$. Using (2.10), we then deduce from the relation $I(\bar{u}) \geqslant I(u^*)$ the following maximum principle (compare [2; Section 1]):

$$\sup_{g_j \in U_j} \int_{\partial B} w_j(\xi, s) g_j(\xi) \, dS_\xi = \int_{\partial B} w_j(\xi, s) \, p_j^*(\xi, s) \, dS_\xi \qquad (j = 1,..., m) \quad (4.2)$$

for almost all s, where

$$w_j(x, t) = \int_B G_j(\xi, T; x, t) \, w_1(\xi) \, d\xi \tag{4.3}$$

and

$$w_1(x) = u^*(x, T) - u_0(x).$$

If we assume that $w_1(x) \not\equiv 0$ and that the uniqueness condition (B') (of Section 2) holds, then it follows from (4.2) that $p^*(\cdot, s) \in \partial U$, i.e., we have a Bang-bang principle. This implies the uniqueness of the optimal solution in case U is strictly convex (which can occur only when $m = 1$).

As shown in [1], if $w_1(x) \equiv 0$ then the Bang-bang principle is false.

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